MATH 1132 Honors
Activity 2 (Ch 7.8 & 6.4)

Name: ______________________________

Improper Integrals and **Escape Velocity**

GOAL We will learn how to apply our knowledge of Improper Integrals and Work to help us compute the velocity that is necessary to escape from Earth's gravitational pull (this is called escape velocity).

BACKGROUND If you stand outside and throw a ball up into the air as hard as you can, the ball will reach the peak of it's trajectory and then come falling back to Earth. If you instead shot an arrow into the air, it would reach a higher height, but would still come plummeting back to the Earth's surface. However, if you were able to launch an object with a high enough velocity, it can overcome Earth's gravitational pull and "escape". Let's dive into the math and physics needed to compute this exact velocity.

When you launch an object, you apply a force to the object in the upward direction. In doing so, we do work on the object and give it energy. Once it leaves the launcher, there is also a force in the downward direction caused by the gravitational pull of the Earth. This gravitational force does work on the ball and tries to take away its energy. The goal is to have enough energy that gravity won't be able to take it all away!

COMPUTING ESCAPE VELOCITY To compute escape velocity, we will need to look at how much work gravity does on our object and then determine how much work we need to do on the object (i.e. how much kinetic energy we need to give it). In the following computations, we will use SI notation (meters, Newtons, Joules, seconds, kg) and will neglect air resistance.

WORK DONE BY GRAVITY

To find the work done by gravity, we start with the force of gravity. In order to determine the gravitational force acting on an object, there is a Universal Law of Gravitation. It states:

$$F_{\text{Gravity}} = \frac{G\, M_E\, m}{(R_E + x)^2}$$

where $G = 6.67 \times 10^{-11}$ Nm^2/kg^2 (Universal Gravitational Constant), $M_E = 5.97 \times 10^{24}$ kg (Mass of the Earth), $R_E = 6.38 \times 10^6$ meters (radius of the Earth), x is the distance above the surface of the Earth (measured in meters), and m is the mass of the object (measured in kg). This same equation can be used for other planets by simply replacing M_E and R_E with the respective measurements of the new planet.

Q: Does $F_{Gravity}$ increase or decrease as an object moves away from the Earth?

As the ball travels upward, the force of gravity is acting on it (and doing “work” to slow it down). Gravity will continue to act on the object even as $x \to \infty$, so in order to determine the total work that gravity does on the object, we will need an improper integral:

$$W_{Gravity} = \int_{?}^{?} F_{Gravity}\, dx$$

where x is the distance from the surface of the Earth and $W_{Gravity}$ is measured in Joules.

Q: Why is the integral set up this way and what are the bounds?

TASK: Use the formula for $F_{Gravity}$ given above to compute this integral in the space below.

WORK FROM THE LAUNCH

Now it is our turn to give our object energy through our initial launch. We can do this by doing "work" on our object and giving it energy (in this case kinetic energy). This starts with applying a force to the object. Newton's 2nd Law of Motion tells us that:

$$F = \text{MASS} \times \text{ACCELERATION}$$

Recall that acceleration $a = \frac{dv}{dt}$ and $v = \frac{dx}{dt}$. Therefore, acceleration can alternatively be written as $a = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx}$ and hence:

$$F = \text{MASS} \times \text{ACCELERATION} = m\,v\frac{dv}{dx}$$

By launching the ball into the air, we are doing work on the ball as we accelerate it from standing still ($v = 0$) to having an initial velocity (v_0) moving it through a small distance Δd (think of that as the barrel of the launcher). We can use Newton's 2nd Law to help us calculate this work:

$$W_{Launcher} = \int_0^{\Delta d} F dx = \int_0^{\Delta d} m\,v\frac{dv}{dx}dx = \int_?^? m\,v\,dv$$

TASK: Determine the bounds and compute this integral in the space below.

NOTE: The formula that we just computed is actually the same formula for the Kinetic Energy of an object of mass m with velocity v_0.

<u>THE TUG -o-WAR:</u>

In order to overpower the work that gravity will enact on the ball, we need to give the ball enough energy from our initial launch. Now that we know how much work gravity will do on the ball and how much energy we can give the ball, we can find our escape velocity.

TASK: Set $W_{Gravity} = W_{Launch}$ and solve for v.

ADDITIONAL QUESTIONS

[**Question 1**] Does it make sense that the escape velocity **does not** depend on the mass of the object being launched? Explain?

[**Question 2**] Convert the units of the escape velocity that you found to MPH.

[**Question 3**] Using your formula for escape velocity, compute the escape velocity on the Moon (Note: you will need to look up important features of the moon). Does your result make sense?

[**Question 4**] You can apply this same formula to any planet or "body". When would you get an <u>infinite</u> escape velocity? What type of planetary object would this be?