SIR Epidemic Models

This is a self-guided group activity. Today we will explore a basic S.I.R. Epidemic model. SIR models are structured models with 3 classes (susceptible, infected, and recovered). Susceptible individuals can become infected. Infected individuals can recover. Recovered individuals stay immune. Deaths can happen within any of the classes and individuals in any class can give birth.

0.1 Model Set-up

S.I.R. Set-up

- $S_t = \#$ of Susceptible Individuals at time t.
- $I_t = \#$ of Infected Individuals at time t.
- $R_t = \#$ of Recovered Individuals at time t.
- $N =$ Total Population
- $b =$ Birth Rate $(0 < b \leq 1)$
- $d =$ Death Rate $(0 < d \leq 1)$
- $\gamma =$ Recovery Rate $(0 < d \leq 1)$
- $\beta =$ Contact Number $(0 < \beta < 1)$
- $\dfrac{\beta S_t I_t}{N} =$ Total number of new infections.

Assumptions: We make the following assumptions for our model:

- We are all born *susceptible*.
- Total population size is constant.
- b and d are constant for all classes (S,I,R).
- Recovered individuals maintain immunity.
- $0 < b + \gamma < 1$

Q1: How does an individual enter the class S_{t+1}? How can it exit?

Q2: How does an individual enter the class I_{t+1}? How can it exit?

Q3: How does an individual enter the class R_{t+1}? How can it exit?

Q4: Draw the *Life-Cycle* graph for this model.

The S.I.R. Model

$S_{t+1} =$

$I_{t+1} =$

$R_{t+1} =$

Q5: Show that if $S_0 + I_0 + R_0 = N$ then $S_t + I_t + R_t = N$ for ALL t (i.e. population remains constant). **Hint:** Find $S_{t+1} + I_{t+1} + R_{t+1}$ and show it equals $S_t + I_t + R_t$.

Note: With a constant population size N, the 3D SIR model can be reduced to a 2D SI model.

Q6: Eliminate R_t from the model above, by using $S_t + I_t + R_t = N$ (i.e. $R_t = N - S_t - I_t$) and inputting to the equations for S_{t+1} and I_{t+1}.

0.2 Equilibrium Points

Let $S \to x$ and $I \to y$ so that our model is of the following form:

$$x_{t+1} = f(x_t, y_t)$$
$$y_{t+1} = g(x_t, y_t)$$

Q6: What are $f(x, y)$ and $g(x, y)$?

Q7: Find all of the equilibrium points of the above 2D system of difference equations. Define the meaning of each point (i.e. eradication, etc) and state the parameter restrictions for which each equilibrium point exists.

0.3 Basic Reproduction Ratios R_0

The endemic equilibrium point that you found in the previous step only exists when a certain expression is > 1. This expression is referred to as the *Basic Reproduction Ratio* and is labelled R_0.

Q8: What is the expression for R_0 with respect to this model (found in previous step)?

The meaning of R_0.

- When $R_0 > 1$ the **Endemic Equilibrium Point** exists. Larger values of R_0 indicate a faster spread (bad for us).
- When $R_0 < 1$ there is only one equilibrium point called the **Disease Free Equilibrium** and this represents eradication of the disease. Smaller values of R_0 indicate a disease that will be eradicated quicker (good for us).
- In practice, R_0 is computed by epidemiologists using real world data related to community spread. The aim is to determine necessary steps to make sure $R_0 < 1$.

Q9: What is the current value of R_0 for the COVID-19 pandemic? What was the highest value of R_0 that was found throughout the pandemic?

Q10: Find R_0 numbers for some of the most serious epidemics that we have seen in history: Scarlet Fever (1910-1920), Measles (1910-1930), Mumps (1912-1916), and Polio (1955).

0.4 Local Stability Analysis

Claim: Disease Free Equilibrium

If $R_0 < 1$ then there exists a unique equilibrium point $(N, 0)$ that is *locally asymptotically stable.*

Q11: Verify the above claim. Hint: compute the Jacobian matrix for this system and finding $\lambda_{1,2}$.

Note: It can be shown that for $R_0 > 1$, the *Endemic Equilibrium Point* is LAS provided that $2 - bR_0 > 0$. We will skip this computation.

SIR Epidemic Model w/ Vaccination

Now, let's suppose we develop a vaccine for this epidemic. The vaccine is highly effective and prevents susceptible individuals from becoming infected. This means that, after taking the vaccine, susceptible individuals enter the recovered class.

S.I.R. w/ Vaccine
Let's make the following changes

- $p =$ Vaccination Rate (proportion of susceptible individuals that get vaccinated during each interval). Note that $0 < p < 1$.
- $0 < p + b < 1$ and $0 < b + \gamma < 1$.

$$\begin{aligned} S_{t+1} &= (1-p)S_t - \frac{\beta}{N} I_t S_t + b(I_t + R_t) \\ I_{t+1} &= \frac{\beta}{N} I_t S_t + (1 - b - \gamma) I_t \\ R_{t+1} &= (1-b)R_t + \gamma I_t + pS_t \end{aligned}$$

Q12: Draw the *life cycle* graph for this new model.

Q13: Explain how vaccination is incorporated into this model (as compared to the Model from without vaccination) and why it makes intuitive sense.

Q14: Eliminate R_t from the model above, by using $S_t + I_t + R_t = N$ (i.e. $R_t = N - S_t - I_t$) and inputting to the equations for S_{t+1} and I_{t+1}. This converts the model to a 2D SI model.

Q15: Find the disease free and endemic equilibrium points for this "SI" model. When does the endemic equilibrium point exist?

Q16: Show that the disease free equilibrium point is LAS if $\mathcal{R}_0 < 1$. What is R_0?

Q17: Suppose that $\beta = 0.5$, $b = 0.05$, $\gamma = 0.05$. Find the minimum vaccination proportion $p_{\min}$ such that $R_0 \leq 1$.